Ex: $$\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3} \text{ "=" } \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{(2x+1)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}} = \lim_{x\to -1} \frac{2x+1}{x-3} = \frac{1}{4}$$

$$\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3} \overset{H}{=} \lim_{x\to -1} \frac{4x+3}{2x-2} = \frac{-4+3}{-2-2} = \frac{1}{4}$$

Def: The following are <u>indeterminate forms</u>

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$

1^∞, ∞^0, 0^0

L'Hôpital's Rule

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).

Suppose that

① $\lim\limits_{x\to a} f(x) = 0$ & $\lim\limits_{x\to a} g(x) = 0$ $\left(\frac{0}{0}\right)$

or

② $\lim\limits_{x\to a} f(x) = \pm\infty$ & $\lim\limits_{x\to a} g(x) = \pm\infty$, $\left(\frac{\infty}{\infty}\right)$

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} \overset{H}{=} \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ (also applies to $x\to a^-$, $x\to\infty$, $x\to a^+$, $x\to -\infty$)

if the limit on the right side exists.

(For notational convenience, write $\overset{H}{=}$ when you use l'Hopital's rule)

Ex:

① $$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4} \text{ "="} \frac{16-8-8}{4-4} \text{ "="} \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 4} \frac{2x-2}{1} = \frac{8-2}{1} = 6$$

② $$\lim_{\theta \to \pi} \frac{1+\cos\theta}{1-\cos\theta} \text{ "="} \frac{1-1}{1-(-1)} = \frac{0}{2}$$

cannot use l'Hôpital's rule

③ $$\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta} \text{ "="} \frac{1-1}{1+(-1)} = \frac{0}{0}$$

$$\overset{H}{=} \lim_{\theta \to \frac{\pi}{2}} \frac{-\cos\theta}{-2\sin 2\theta} \text{ "="} \frac{0}{0}$$

$$\overset{H}{=} \lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{-4(-1)} = \boxed{\frac{1}{4}}$$

④ $\lim_{x\to\infty} \dfrac{x+x^2}{1-2x^2}$ "=" $\dfrac{\infty}{-\infty}$

$\overset{H}{=} \lim_{x\to\infty} \dfrac{1+2x}{1-4x} \overset{H}{=} \lim_{x\to\infty} \dfrac{2}{-4} = \boxed{\dfrac{-1}{2}}$

same

$\lim_{x\to\infty} \dfrac{x+x^{2}}{1-2x^{2}} = \boxed{\dfrac{1}{-2}}$

$\underline{Ex}$: $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ "=" $\boxed{\infty}\cdot 0$

0 "=" $\dfrac{1}{\infty}$

∞ "=" $\dfrac{1}{0}$

$= \lim_{x\to\infty} \dfrac{1}{\frac{1}{x}} \cdot \sin\left(\frac{1}{x}\right)$

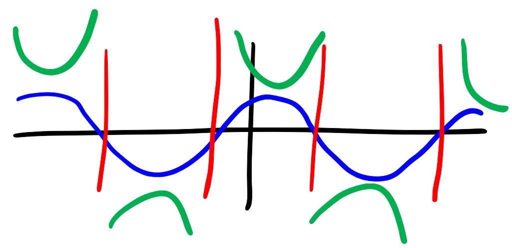
$= \lim_{x\to\infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$ "=" $\dfrac{0}{0}$

$\overset{H}{=} \lim_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = \boxed{1}$

If we have the indeterminate form $\infty \cdot 0$ use the fact that

$$f \cdot g = f \cdot \frac{1}{\frac{1}{g}} = \frac{1}{\frac{1}{f}} \cdot g$$

$$= \frac{f}{\frac{1}{g}} = \frac{g}{\frac{1}{f}}$$



Ex: $\lim\limits_{x \to \frac{\pi}{2}^-} \underbrace{\cos x}\cdot \underbrace{\sec 5x}$ "=" $0 \cdot \infty$

$$= \lim_{x \to \frac{\pi}{2}^-} \cos x \frac{1}{\frac{1}{\sec 5x}} = \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{\cos 5x}$$

Ex: $\lim\limits_{x \to \infty} x^3 e^{-x^2}$ "=" $\infty \cdot 0$

$$= \lim_{x \to \infty} \frac{x^3}{e^{x^2}}$$